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THEORY OF MULTIPLICATIVE PROCESSES. I.

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ABSTRACT

General properties of statistics of multiplicative systems are discussed together with the study of fluctuations in the number of particles in such systems. A general method is indicated through which one may study the fluctuations in the case where one takes into account the factors of geometry and time-dependence of constants.

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THEORY OF MULTIPLICATIVE PROCESSES. I.

The statistical theory of multiplicative chain processes does not compare in completeness to date with the corresponding theory of additive processes. The present paper is intended primarily as an exposition of a simple theory of the statistics of multiplication, permitting application to a variety of special problems.

The simplest (the "Bernoullian") case may be described as follows: A particle can produce, with probabilities p_0, p_1, p_2 ..... p_n .... a number 0, 1, 2, 3, ..., n, ... of similar particles in one generation. We assume that each particle produced has again the same probabilities of producing n offspring. We also assume that each particle dies at procreation. Required is the probability law $p_k(n)$ for any generation k.

We remark parenthetically that this formulation makes the multiplicative process essentially discrete and finite. The statistics of neutron multiplication involves a continuous process as well, namely a random distribution in energy, space and time. We disregard this aspect initially. Later we shall show that the admission of such continuity leads to a generalization of the methods described below. There are, in the meantime, two physically accurate interpretations of a discrete series: (1) one can represent the chain process as a graph; the n particles in the kth generation are the n lines connecting the kth - and the k + 1st - branch points in a chain or set of chains; (2) the n particles are those in existence at the kth unit of time, where the probability law $p_1(n)$ is the distribution one unit of time after the introduction of a single particle.

If a time unit be chosen equal to the average time between fissions, the distinction is in many cases not crucial. Frankel, and later Feynman studied the continuous process. We shall show later that their differential equations of the random process correspond to the infinitesimal transformations of the group in which our iteration (see Th. I) may be imbedded.

1. The first problem to consider is this: We are given an amount and arrangement of active material. In this system a neutron produces on the average n neutrons with probability $p(n)$; $\sum_{n=0}^{\infty} p(n) = 1$. $p(0)$ is the average probability of leakage or absorption, without subsequent production of neutrons. $p(n)$ normalized for $n > 0$ is a nuclear constant, so far purely empirical, known as to its first moment and less accurately as to its second moment. Required is the probability of having n neutrons after k generations (or units of time). This problem is solved, in principle, by:

Theorem I. Let $f(x)$ be the generating function of the distribution of the number of offspring, i.e., $f(x) = \sum_{n=0}^{\infty} p(n) x^n$. Then the generating function for the kth generation $f_k(x) = f^k(x)$, the kth iterate of $f(x)$. [The kth iterate is defined as follows: $f^1(x) = f(x)$, $f^k(x) = f(f^{k-1}(x))$]. The theorem asserts that the probability $p_k(n)$ is given as the coefficient of x^n in the ascending polynomial or power series expression of $f^k(x)$. The physical multiplication of the random variable is reflected in the iterated substitution by which $x \to f(x)$.

Proof: Starting with one neutron in the 0th generation we obtain, with probability $p_k(n)$, n neutrons in the kth generation. Beginning with r neutrons, denote the corresponding probability by $p_k^{(r)}(n)$. Now assume that

a chain is started by one neutron. We have

$$p_k(n) = \sum_{r=0}^{\infty} p_{k-1}(r)\, p_1^{(r)}(n)$$

Now if $f(x)$ is the generating function of the distribution $p_1(n)$, the generating function of the distribution $p_1^{(r)}(n)$ is $\left[f(x)\right]^r$. This follows from the assumption that contemporary neutrons are independent in procreative powers, and from the theorem (of Laplace) that the generating function of a sum of independent random variables is the product of their generating functions. The above proposition may also be verified for $r = 0$, since $p_1^{(0)}(n) = 0$ for all $n > 0$. Substituting generating function for probability in the above equation, we have:

$$f_k(x) = p_{k-1}(0) \cdot \left[f(x)\right]^0 + p_{k-1}(1)\left[f(x)\right]^1 + p_{k-1}(2)\left[f(x)\right]^2 + \ldots$$

$$+ \ldots = f_{k-1}\left[f(x)\right] = f^k(x).$$

Two remarks may be made at this point. (a) The simple proof above sustains a more general theorem if the distribution generated by $f(x)$ is not constant, but time- or generation-dependent. Instead of the iterate $ffff\ldots\left(f(x)\right)$, we will have some $fgh\ldots\left(q(x)\right)$. By the mode of argument established, the chain process may be analyzed one step further. Let $g(y) = ay + b$ be the generating function for the probabilities b of loss or absorption of a single neutron and a of producing fission, with $a + b = 1$. Let $h(x) = c_1 x + c_2x^2 + c_3x^3 + \ldots$ be the generating function of distribution of neutrons per fission. Then if the two are combined by the transformation $y \to h(x)$, we have that the distribution of neutrons per neutron is generated by $f(x) = g\left[h(x)\right]$. If on the other hand we start from a single fission, and wish to know the distribution for the number of first-generation

missions, this is given by $F(y) = h(g(y))$. The iterates of $f(x)$ and $F(y)$ are connected by simple and evident relations.

There remains the practical problem of determining coefficients and other properties of $f^k(x)$, given $f(x)$. To this end we first shall establish some general properties of iteration.

2. Let $f(x)$ be a monotone function. Assume e.g. $f(x)$ increasing, i.e., if $x < y$, $f(x) < f(y)$. A fixed point for $f(x)$ is a value x_0 such that $f(x_0) = x_0$. The set of fixed points for a continuous function is closed, i.e., the points which are not fixed form a collection of disjoint intervals, whose endpoints are fixed points. If we form the sequence $f^k(x)$ for a given x we obtain a sequence of points converging to a fixed point x_0 which forms the endpoint of the interval in which x is situated. In fact, there are two cases possible, either $f(x) < x$ or $f(x) > x$. From the monotone character of $f(x)$ it follows that we shall have correspondingly either $f^k(x) < f^{k-1}(x)$ or $f^k(x) > f^{k-1}(x)$ for all k. Unless these sequences tend to $-\infty$ or $+\infty$, they will have limit points. If now $\lim_{k=\infty} f^k(x) = x_0$, we must have $f(x_0) = x_0$. In fact $\lim_{k=\infty} f(f^k(x_0)) = f(x_0) = x_0$. In addition, it is easy to see that x_0 is the next fixed point to x (on the left or right depending on whether $f(x) < x$ or $f(x) > x$. This follows from the fact that if $f(x)$ is monotone and $f(x_0) = x_0$, $f(x_1) = x_1$, then for all x such that $x_0 < x < x_1$, we have $f(x_0) = x_0 < f(x) < f(x_1) = x_1$.

In our case $f(x)$ is a power series with all coefficients non-negative, $f(0) \geq 0$, $f(1) = 1$. This function is certainly monotone and increasing for all non-negative x. Let x_0 be the first (non-negative) fixed point. x_0 certainly exists, the set of fixed points being closed. From

these conditions it follows that $\lim_{k=\infty} f^k(0) = x_0$. But if the variable in a generating function is set = 0, the value of the function is the probability that the random variable takes the value 0. Hence x_0 gives us the limit of the probability of mortality in the system. The probability of immortality is therefore simply $1 - x_0$, where x_0 is the smallest non-negative root of the equation $f(x) = x$. It is easy to see that if, as in our case, all the coefficients in the expansion of $f(\ )$ are non-negative and $f(1) = 1$, then from $f'(1) > 1$ it follows that there is a root, and only one root x_0 which is non-negative and <1. If $f'(1) \leq 1$, $x_0 = 1$ is the smallest positive root. We obtain immediately therefore the familiar fact that neutrons in a subcritical gadget without source will, with probability 1, die out in a finite time. For the supercritical gadget the probability of indefinite production can be obtained by solving the equation $f(x) = x$.

The kth iterate of a function can be obtained by a simple graphical or mechanical method which is based on the fact that along the diagonal, $f(x) = x$. Thus we may for given x replace this argument by $f(x)$, getting $f^2(x)$ graphically, then repeating $f^4(x)$ and so forth. In the case of the generating function under discussion this shows that $f^k(x)$ very rapidly approaches its asymptotic form: for the critical or subcritical case the asymptote in the interval $0 \leq x < 1$ is $\lim_{k=\infty} f^k(x) \equiv 1$; for the supercritical case in the interval $0 \leq x < 1$ the asymptote is $\lim_{k=\infty} f^k(x) = x_0$. This implies that for all positive powers of x in $f^k(x)$ the coefficients approach 0 uniformly, i.e., the mass of probability is either absorbed altogether into the zero region (subcritical case), or is spread out in an infinitely long tail (supercritical case). In the region of criticality the

distribution has an infinitely long tail with mass approaching zero as the probability of mortality approaches one.

3. One of the important properties of generating functions is that they permit the calculation of moments. Thus if p_n is the distribution itself, $f(x) = \sum_{n=0}^{\infty} p_n x^n$ its generating function, we have, because obviously $f(1) = 1$, the first moment or expected value of the random variable $= \sum_{n=0}^{\infty} n\, p_n = f'(1)$ = the first derivative of $f(x)$ at $x = 1$. Similarly the second moment of the number of neutrons can be found if we know the second derivative.

In fact

$$\sum_{n=0}^{\infty} n^2 p_n = \sum_{n=0}^{\infty} n(n-1)p_n + \sum_{n=0}^{\infty} np_n = f''(1) + f'(1)$$

Similarly the rth moment can be found easily from the values of the first r derivatives of $f(x)$ at $x=1$. (The rth derivative at $x = 1$ is sometimes called the rth combinatorial moment.)

Our generating function is the kth iterate $f^k(x)$. It turns out that its first m derivatives depend only on the first m derivatives of $f(x)$ itself in a rather simple way. We have, in fact:

Theorem II. (a) $\left[f^k(x)\right]'_{x=1} = f'(1) = \bar{\nu}^n$

(i.e. the proof of the intuitively obvious result that the expected number of neutrons after n generations is $\bar{\nu}^n$).

(b) $\left[f^k(x)\right]''_{x=1} = f''(x)_{x=1} \cdot \left[\left(f'(1)\right)^k + \left(f'(1)\right)^{k+1} + \ldots \left(f'(1)\right)^{2k-1}\right]$

The proof is immediate by induction:

$$\left[f\left(f^{k-1}(x)\right)\right]' = f'\left(f^{k-1}(x)\right) \cdot \left[f^{k-1}(x)\right]'$$

But for $x = 1$, $f^{k-1}(x) = x = 1$; therefore since by assumption $\left[f^{k-1}(x)\right]'_{x=1} = \left[f'(1)\right]^{k-1}$ we obtain our formula (a). By differentiating twice we obtain (b). Somewhat more complicated formulae hold for higher

derivatives:

Their derivation is through recursive relations as follows:

by differentiating the identical equation

$$f^k(x) = f\left(f^{k-1}(x)\right)$$

repeatedly, and in all places substituting x_o for $f^{k-1}(x_o)$, we obtain a sequence of linear first-order difference equations. Representing

$\frac{d^r}{dx^r} \cdot f^k(x) = M_{k,r}$ $(M_{1,r} = M_r)$ we obtain

$$M_{k,1} = M_1 \cdot M_{k-1,1}$$

$$M_{k,2} = M_2 \cdot M^2{}_{k-1} + M_1 \cdot M_{k-1,2}$$

$$M_{k,3} = M_3 \cdot M^3{}_{k-1,1} + 3 M_2 M_{k-1,1} \cdot M_{k-1,2} + M_1 M_{k-1,3}$$

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each is of the form

$$x_k = A_{k-1} + M_1 x_{k-1}$$

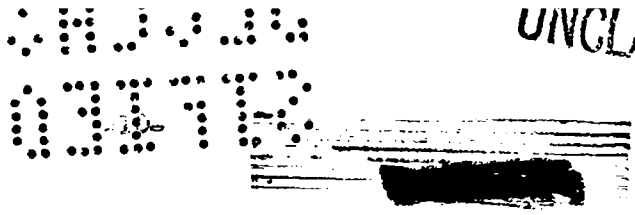
whose general solution is

$$x_k = \sum_{S=2}^{k} M_1{}^{k-s} A_{s-1} + M_1{}^{k-1} x_1$$

Solutions, for the first three derivatives are

$$M_{k,1} = M_1{}^k$$

$$M_{k,2} = M_2 \cdot M_1{}^{k-1} \left[\frac{1 - M_1{}^k}{1 - M_1}\right]$$

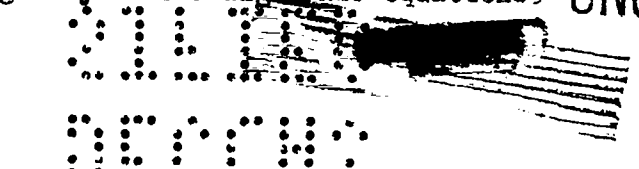


$$M_{k,3} = M_3 M_1^{k-1}\left[\frac{1 - M_1^{2k}}{1 - M_1^2}\right] + \frac{3 M_2^2 M_1^{k-1}}{(1 - M_1)}\left[\frac{1 - M_1^{k-1}}{1 - M_1}\right]$$

$$- \frac{3 M_2^2}{(1 - M_1)} M_1^k \left[\frac{1 - M_1^{2k-2}}{1 - M_1^2}\right]$$

Since in the function under discussion $x_0 = 1$ is a fixed point, these derivatives are the combinatorial moments of the distribution. We may now consider the three cases where M_1 ($= \bar{\nu}$ of common use) is >1, <1, or $=1$.

a) In the supercritical case where $M_1 > 1$, it is clear from the method of deriving these factorial moments that if the random variable n is measured in units of M_1, all moments approach a finite asymptotic form. Computation of moments for this asymptotic distribution may be greatly simplified as follows: Let us define a function $\phi(x) = x^{1/M_1}$, the kth iterate being $\phi^k(x) = x^{(1/M_1)^k}$. The generating function $f^k\left(\phi^k(x)\right)$ if expanded in powers of $x^{(1/M_1)^k}$ has the same coefficients as $f^k(x)$ but these are now probabilities associated with the number of particles measured as fractions of the expected number. This is to say that the distribution is scaled in units of $M_1^k = \bar{\nu}^k$, and its first moment = 1. Since for the supercritical case all moments approach a constant value as $k \to \infty$ when scaled in this way, and since the generating function is monotonic in the region $(0,\infty)$, there exists a common limiting value, $g(x)$ of both $f^k\left[\phi^k(x)\right]$ and $f^{k-1}\left[\phi^{k-1}(x)\right]$. Since $f^k\left[\phi^k(x)\right] = f\left[\phi^{k-1}\left[\phi^k(\phi(x))\right]\right]$, we may write in the limit: $g(x) = f\left[g\left[\phi(x)\right]\right]$; $\phi(x) = x^{1/M_1}$, $f(x)$ given, and from this functional equation for g, its moments may be obtained from the second, third, etc. derivatives of g by solving only linear algebraic equations.



b) In the exactly critical case, $M_1 = 1$, the moments are

$$M_{k,1} = 1$$
$$M_{k,2} = k \cdot M_2$$
$$M_{k,3} = k M_3 + 3 \binom{k}{2} M_2^2$$

This is a distribution in which $P_{k,o} \cong 1 - 1/kM_2$, and such that if the system has not died in the kth generation, the expected number of neutrons is $\cong kM_2$.

c) In the subcritical case all moments converge to zero, but are approximately proportional to the first moment.

4. We may consider here briefly a simple special case, in which the iteration problem may be solved exactly.

Let $f(x) = (ax + b)/(cx + d)$; we have here a three-parameter family of functions (one of the four constants a, b, c, d, is immaterial). We can adjust them so that $f(1) = 1$, and $f'(1) = \overline{\nu}$. We can then impose another condition, either on $f''(1)$, or so that $f(x_o) = x_o$, where x_o is the "true" probability of mortality. Functions of the above sort form a group under substitution. This can be verified directly by substituting. (They form the so-called projective group of the line.) A fortiori the iterated function

$$f^k(x) = (a_k x + b_k)/(c_k x + d_k)$$

By expanding $f^k(x)$ in a power series in x, we obtain the exact solution of our problem in this fairly general case. We determine the constants by the following three relations:

(1) Because $f(1) = 1$, we have for every k: $f^k(1) = 1$ which gives

$$a_k + b_k = c_k + d_k$$

(2) Similarly, for the second fixed point x_o of $f(x)$, i.e., the root

$x_o \neq 1$ of $f(x) = x$, we have $f(x_o) = x_o$, and therefore for all k: $f^k(x_o) = x_o$ or $a_k x_o + b_k = c_k x_o + d_k$ and $x_o = -b/c$, from $ax_o + b = x_o(cx_o + d)$.

(3). From the results of section 3, we know that

$$\left[f^k(x)\right]'_{x=1} = \left[f'(1)\right]^k = \bar{\nu}^k$$

This gives

$$\left.\frac{a(cx+d) - (c\,ax + b)}{(cx+d)^2}\right|_{x=1} = \bar{\nu}$$

or taking account of (1)

$$(a - c)/(c + d) = \bar{\nu}$$

and therefore for all k

$$(a_k - c_k)/(c_k - d_k) = \bar{\nu}^k$$

From the above three relations it is easy to calculate the constants a_k, b_k, c_k, d_k in terms of $\bar{\nu}$ and one arbitrary parameter. By eliminating a_k, b_k and developing into a power series, we get, noting that $c_k/d_k = 1/\bar{\nu}^k - 1$, assuming e.g. $\bar{\nu} > 1$, the result in the form

$$f^k(x) = \left[(Ax + B)/\bar{\nu}^k + 1\right]\left\{1 + (1 - 1/\nu^k)x + (1 - 1/\nu^k)^2 x^2 + \cdots (1 - 1/\nu^k)^n x^n + \cdots\right.$$

This constitutes a complete solution of our problem. It is interesting to note that the probability of having n neutrons decreases geometrically with n; the ratio of the successive terms is in the case $\bar{\nu} > 1$, k large extremely close to 1. The distribution has the form of an exponential, decreasingly very slowly. Asymptotically the probability of having exactly n neutrons is independent of n! This result shows also the possibility of enormous fluctuations in multiplicative systems. The "law of large numbers"

in its ordinary formulation is not true for multiplicative processes. In fact the probability of having more (or less) than ℓ times the expected value of neutrons tends to a positive constant (dependent on ℓ). The following form of the law of large numbers is valid, as the examination of the distribution shows at once:

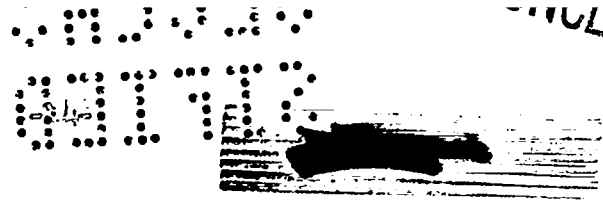
Theorem III. Given an $\epsilon > 0$, there exists an N such that for all $k > N$, the probability of the number n of neutrons in the kth generation being such that $(\bar{\nu} - \epsilon)^k < n < (\bar{\nu} + \epsilon)^k$ is greater than $1 - \epsilon$:

$$P\left\{(\bar{\nu} - \epsilon)^k < n < (\bar{\nu} + \epsilon)^k\right\} > 1 - \epsilon$$

It remains to discuss the most general form of the distribution. We hope to do this in part II of the paper through two methods, one consisting of the consideration of functions of the form $h\, f\, h^{-1}(x)$, where f is of the projective linear form discussed above, and h(x) is an arbitrary monotonic function. The kth iterate then is simply $h\, f^k\, h^{-1}(x)$. The function h(x) will give us more arbitrary parameters for our real distribution. The second method consists in developing f(x) into a series of functions whose terms have the "projective" form.

Finally it may be remarked that the limiting distribution obtained above is formally identical to those obtained by Frankel (LAMS-36) and Feynman who used a continuous time parameter instead of our discrete-generations model. Their physical model is somewhat different and leads to the finding of the infinitesimal transformation of the continuous, abelian, one-parameter group into which the group of iterates of a function can be imbedded.

5. There are many other problems besides the question of the probable number of neutrons after k generations which may be solved by operational methods.



The first we shall consider is that of a subcritical system ($\bar{\nu}<1$) with a source. We suppose that the distribution of neutrons entering the system in a given generation has the generating function $\phi(x)$. $f(x)$ being the generating function of the system itself as before, we shall have

Theorem IV. The generating functions in the zero, first, second generations are the functions:

$$\phi(x),\ \phi(x)\cdot\phi\left[f(x)\right],\ \phi(x)\cdot\phi\left[f(x)\right]\cdot\phi\left[f^2(x)\right]$$

Proof is completely analogous to that of theorem I.

In general, letting $F_k(x)$ represent the distribution in the kth generation

a) $$F_k(x) = \phi(x)\cdot F_{k-1}\left[f(x)\right]$$

If the system is subcritical, but sustained at a definite level by the source, we shall have the limiting distribution - or its limiting generating function - as a nonsingular function of x: $\lim_{k\to\infty} F_k(x) = F(x)$, $F(1) = 1$. Passing to the limit on both sides of our equation a) we get

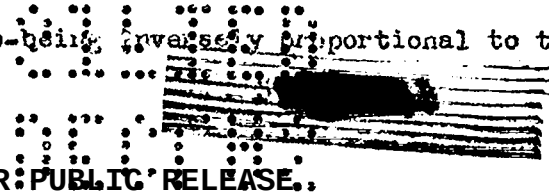
b) $$F(x) = \phi(x)\circ F\left[f(x)\right]$$ where $\phi(x)$, $f(x)$ are given.

One has to determine $F(x)$ from this functional equation. Even without doing it one can obtain at once useful statistical information, for example the moments of $F(x)$, by differentiating (b). Thus:

$$F'(1) = \phi'/(1-f')$$

$$F''(1) = \frac{\phi''}{(1-f'^2)} + \frac{(2\phi' f' + f'')\phi'}{(1-f'^2)(1-f')}$$

giving us a way to compute standard deviations, and similarly, more complicated expressions for the higher derivatives and moments. The first derivative -- the expected value-being inversely proportional to the degree



of subcriticality becomes infinite if $f'(1)$ approaches 1.

6. We come now to the probability distribution of the sum of all neutrons in the system from the first to the kth generation. We have established previously that if $f(x) = \sum_{n=0}^{\infty} p_n x^n$ is the generating function for the probabilities of n particles in the first generation then the generating function of the kth generation is given by the kth iterate $f^k(x)$.

If we want the generating function for probabilities of having the total of n particles from the first to the kth generation, we shall proceed as follows.

The total of n particles can be obtained by any one of the following mutually exclusive cases: we can have 1 in the first generation and $n - 1$ in the remaining $k - 1$, or 2 in the first generation and $n - 2$ in the remaining $k - 1$, in general we can have r in the first and $n - r$ in the remaining $k - 1$ generations. The required probability is therefore the sum of

$$q(n) = \sum_r p_r \cdot p_{n-r}^{k-1}(n)$$

Here $p_{n-r}^{k-1}(n)$ denotes the probability that, starting from r in the first generation, we shall attain from these r a total of $n - r$ in $k - 1$ generations. But the r particles are independent of each other. The probability of getting the total of $n - r$ from them is therefore the probability of $n - r$ in the sum of these r variables. The generating function for the sum of the independent variables is the product of the generating functions corresponding to each of them. In our case it is the rth power of $f(x)$. We are looking for the coefficient of x^{n-r} in $\left[f^{k-1}(x)\right]^r$. Our required probability q_k equals therefore the sum with respect to r of coefficients of x^{n-r} in $\left[f^{k-1}(x)\right]^r$, or the sum of the

coefficients of x^n in $\sum_r p_n x^r \circ \left[f^{k-1}(x)\right]^r$.

But the coefficient of x^n in $\sum_r p_n x^r \left[f^{k-1}(x)\right]^r$ is the same as this coefficient in $f\left(x \circ f^{k-1}(x)\right)$. This is true for all n. Therefore the generating function for q_n is $f\left(xf^{k-1}(x)\right)$. Since n here is arbitrary we get:

Theorem V. The generating function for the time sum is:

$$u^k(x) = f\left[xu^{k-1}(x)\right]$$

If we "count" the original particle, this multiplies the generating function by x; expressing this slightly modified form recursively, we obtain the more convenient expression

$$u^k(x) = xf\left[u^{k-1}(x)\right]$$

As we know we have, in general, a relation between moments of the nth order of a distribution function and the nth derivative of the generating function. We shall now show how one can compute the derivatives of $u^k(x)$ for any k in an explicit manner.

Since, as was shown above,

$$u^k(x) = xf\left[u^{k-1}(x)\right]$$

we may obtain the desired results by repeated differentiations, and by solving the resulting finite difference equations. But if k is allowed to approach infinity, and if the system is subcritical,

$$\lim_{k\to\infty} u^k(x) = \lim_{k\to\infty} u^{k-1}(x) = u(x)$$

Hence for the distribution of the total number produced, we have

$$u(x) = x \circ f\left[u(x)\right]$$

differentiating, we obtain:

$$u'(1) = 1/\left(1 - f'(1)\right)$$

$$u''(1) = \left[f'' + f'\left(1 - f'\right)\right] / \left(1 - f'\right)^3$$

These examples show how moments of the distributions can be computed for various problems in our discrete model. Otto Frisch has shown, how, from the form of these moments one can write their correct form for the continuous model, without having to solve the partial differential equations of the problem. This correspondence between the two models will be taken up in part II. It may be said that a generality of method has been established by the foregoing results, which demonstrate that the iteration of suitable operators corresponds to various physical observables connected with chain processes. For example it may be mentioned that the transformation $x \to (1/x)\ f(x)$ gives us the probability-distribution for differences between the number of neutrons in a generation and the number in the next generation. Thus $f^{k-1}\left[(1/x) \circ f(x)\right]$ generates probabilities of this kind. The mathematical description of a multiplicative chain process is seen to involve the iteration of a functional operator U. These operators U act on the domain of all monotone functions $g(x)$, $g(1) = 1$. To summarize again just a few examples:

(1). $U(g) = f(g)$, f here is a given monotone function, g represents any function of the domain on which U operates, i.e., $g(x)$ monotonic, $g(1) = 1$. This operator U is the only one that has been studied extensively in literature. Its iteration leads to the simple iteration process:

$$g(x),\ f(g(x)),\ f\left[f\left(g\ (x)\right)\right] \cdot \cdot \cdot \cdot f^k(g(x)) \cdot \cdot \cdot \cdot .$$

(2) $U(g) = f(x \circ g)$, f a given function

The domain of the operator, i.e., the admissible g are the same, but there seems to be very little known about the iterates of this operator. This operator is tied to the probability law of the total number

of particles produced.

(3) $U(g) = \phi(x) \cdot g(f(x))$; $\phi(x)$, $f(x)$ are given. The iterates of this operator give us the distribution of the number of particles produced when a source with given distribution $\phi(x)$ is acting constantly.

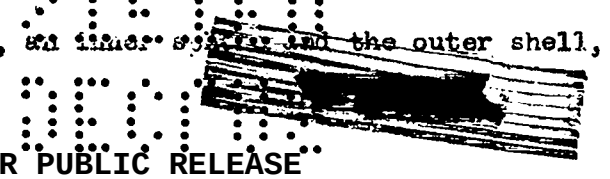
(4) $U(g) = f[(1/x) \cdot g]$

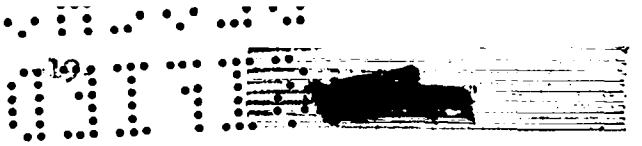
This operator relates to the probability distribution of the difference of the number of particles in successive generations. The study of conjugates, fixed points, etc. for such operators seems to be important. We hope to undertake this study in part II of this paper.

We turn now to a more complex version of the problem. Hitherto it has been assumed that the generating function was independent of temporal and geometrical factors. However, our methods are extensible beyond these limitations.

(7) The calculation of the probability distributions in the general case of heterogeneous particles will now be considered. So far we have assumed that the probability of generating m neutrons is the same independently of the parent neutron. If one takes the real situation where the system of the active material is of finite extent, then obviously the probability of leakage and absorption is a function of position of the parent nucleus. It is obvious that in general chemical or nuclear chain-reaction processes one has to deal with several kinds or even a continuous variety of the elementary generating functions.

In order to explain our methods of iteration of functional operators for this general case we shall take the simplest case of two kinds of particles. If we divide, for the first approximation, the sphere of the active material into two parts, an inner sphere and the outer shell, we shall





characterize the neutrons generated in the one part by the subscript x, the others by subscript y. An x-particle can generate either x-particles again or penetrating to the outer shell y-particles, or, of course, leak out or be absorbed; the same, though with different probabilities, applies to the y-particles. In reality we should consider a one-dimensional variety of kinds of particles corresponding to all values of their distance r from the center of the sphere or even a two-dimensional one if we want to take into account different velocities. To simplify the presentation we shall limit ourselves here to just two kinds (x and y).

We assume that the following elementary probabilities are given by the nuclear constants and by the integrals of the geometry involved.

An x-particle can produce $n(>0)$, x-particles with the probabilities p_n and $n(>0)$ y-particles with probabilities q_n. The probability of dying out - absorption or leakage - will be denoted by p_o.

For the y-particles the corresponding probabilities will be denoted by $\bar{p}_n$, $\bar{q}_n$, and $\bar{p}_o$. It is because of the geometry of the system that $\bar{p}_o$ and p_o are certainly different.

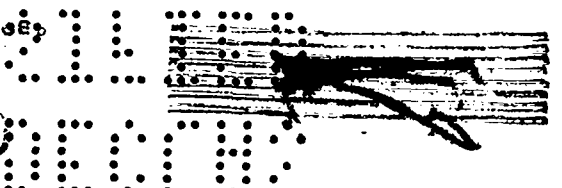
We now write the two functions of two variables each:

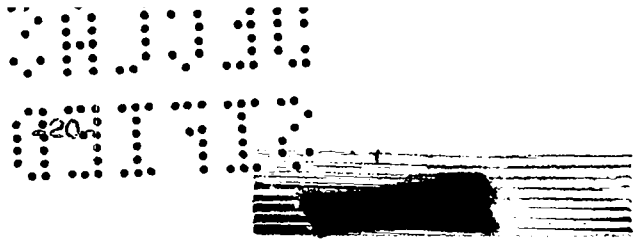
$$f(x,y) = p_o + p_1 x + \ldots p_n x^n + \ldots + q_1 y + \ldots q_n y^n + \ldots$$

$$g(x,y) = \bar{p}_o + \bar{p}_1 x + \ldots \bar{p}_n x^n + \ldots + \bar{q}_1 y + \ldots \bar{q}_n y^n \ldots$$

The coefficients of $f(x,y)$ give the probabilities of having in the first generation a given number of x- or y- particles starting with one x-neutron. Those of $g(x,y)$, if we start with a y-neutrons.

Required are the probabilities of finding in the next generation a given number of x-and y-particles.





Let us form the function

$$f_2(x,y) = f\left[f(x,y),\ g(x,y)\right]$$

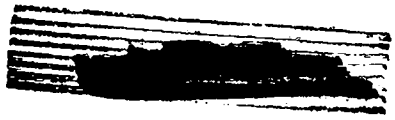
By reasoning exactly as in the proof of Theorem I (or Theorem III) we see that the probability of having n x-particles and m y-particles is given by the coefficient of $x^n y^m$ in $f_2(x,y)$. If we started in 0th generation with a y-particle we will get these probabilities as the coefficients of $x^n y^m$ in $g[f(x,y),\ g(x,y)]$. By an obvious induction we obtain:

Theorem VI. The probabilities of having n x-particles and m y-particles in the kth generation are given by the coefficient of $x^n y^m$ in $f\left[T^{k-1}(r)\right]$ or $g\left[T^{k-1}(r)\right]$ (depending on whether we started from an x- or from a y-particle). $T^k(p)$ is a transformation of the plane (x,y) into itself defined as follows: if $p = (x,y)$ then $T^1(p) = T(p) = \left[f(x,y),\ g(x,y)\right]$; $T^k(r) = T\left(T^{k-1}(p)\right)$

Without going into the details of the proof or actual computations of moments we wish to conclude by the following remarks:

(1). In the case of 3 or any finite number r of different kinds of particles, the formalism necessary to obtain the generating function for the kth generation is the same. It consists of iterating a given set of r functions or a transformation in r dimensions (variables $x_1, x_2 \cdots x_r$).

(2). One fairly general case where the coefficients of the mixed powers of the variables $x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ can be computed explicitly in a closed form for any number k of generations is when the given transformations is the r dimensional generalization of our projective transformations on the line i.e.



$p = (x_1, x_2 \cdots x_r)$; $p' = T(r) = (x'_1\ x'_2 \cdots x'_r)$ where

$$x'_1 = f_1(x_1 \cdots x_r) = (a_{11}x_1 + \cdots a_{1r}x_r + b_1)/(c_{11}x_1 + \cdots c_{1r}x_r + d_1)$$

. . . . . .

. . . . . .

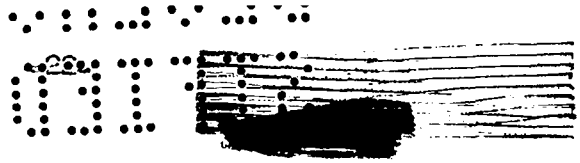
$$x'_r = f_r(x_1 \cdots x_r) = (a_{r1}x_1 + \cdots a_{rr}x_r + b_r)/(c_{r1}x_1 + \cdots c_{rr}x_r + d_r)$$

(3) The computation of moments of the distribution in the most general case does not involve the explicit knowledge of $T^k(r)$, but can be obtained through the knowledge of the moments of the r given elementary functions

$$f_1(x_1 \cdots x_r) \cdots f_r(x_1 \cdots x_r)$$

The role of the numerical multiplication of moments is here taken over by matrix multiplication

(4). The other operators corresponding, e.g., to $U(g) = f(x \cdot g)$, etc., have not been so far investigated in the r-dimensional case.



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CONCLUSIONS REGARDING APPLICATIONS

The expected value of the number of neutrons per fission "ν" is known with fair accuracy. The critical mass and the expected number of neutrons in a gadget depend on this constant alone. Very little seems to be known, however, about the distribution function of the number of neutrons or even only about its second moment. The great fluctuations in multiplicative systems discussed above are of some practical interest for the following reasons:

1. The correct timing of the initiation of the gadget is vital for high efficiency. Even with good sources there will be an uncertainty of several generations time - due to fluctuations in multiplication.

2. The fluctuations of multiplication are of interest in all "integral" experiments.

3. For gadgets large in comparison with the mean free path for fission, the spatial fluctuations may destroy the initial spherical symmetry.

In dealing with such problems it is useful to develop a uniform technique for describing the statistics of multiplicative phenomena. This paper constitutes a first step consisting essentially in the observation that the iterated substitution (of a function, or more generally of a functional operation) represents exactly the statistical laws of multiplicative processes. In the sequel, it is hoped to apply this technique to the study of the problems of geometrical and time-dependence of the process.

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